Lissajous figures*

Lab 1: Introduction to Instrumentation

ECE 209: *Circuits and Electronics Laboratory*

A **Lissajous ("LEE-suh-zhoo") figure** is a parametric plot of the harmonic system

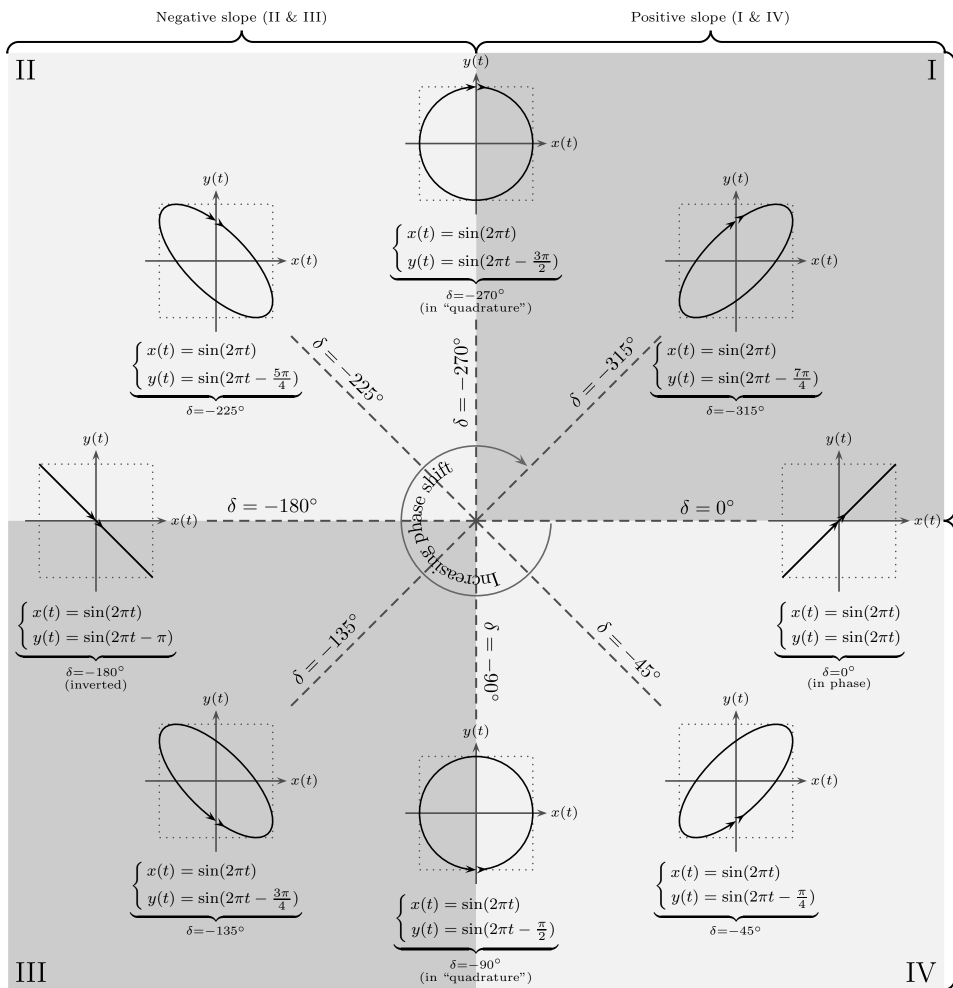
$$\begin{cases} x(t) = A_x \sin(\omega_x t + \phi), \\ y(t) = A_y \sin(\omega_y t + \phi + \delta) \end{cases} \qquad \text{(i.e., } y(x) = A_y \sin\left(\overbrace{\frac{\omega_y}{\omega_x}\left(\arcsin(\tfrac{x}{A_x}) - \phi\right)}^{\omega_y t} + \phi - \delta \right) \text{ where } |x| \leq A_x\text{).}$$

In our case, we plot an input $x(t)$ and output $y(t)$ of a linear time-invariant (LTI) system. Because complex exponentials are eigenfunctions of LTI systems and sinusoids are sums of complex exponentials, the output frequency will match the input frequency (i.e., $\omega_x = \omega_y = \omega = 2\pi f$). Our LTI system (i.e., the phase-shifter circuit) is an all-pass filter, and so it ensures that $A_y = A_x = A$. So we are consider the simpler system

$$\begin{cases} x(t) = A \sin(2\pi f t + \phi), \\ y(t) = A \sin(2\pi f t + \phi + \delta) \end{cases} \qquad \text{(i.e., } \underbrace{y(x) = A \sin\left(\arcsin(\tfrac{x}{A}) - \delta\right)}_{x\text{-}y \text{ graph has no dependence on } \phi.} \text{ where } |x| \leq A\text{),} \tag{1}$$

and we use a Lissajous figure to find the phase shift δ. We obtain the Lissajous figure with the oscilloscope in its X-Y mode with the input of our system tied to the X channel and the output tied to the Y channel. At each instant, the scope plots a dot with the input X sample as the horizontal coordinate and the output Y sample as the vertical coordinate. Because the dots persist on the screen for a short time, their "ghosts" form a Lissajous figure on the screen. To see the rotation direction, we can slow down the input frequency.

LTI Lissajous figures are **ovals** with *eccentricity* and *direction of rotation* determined by phase shift δ.



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*Document from http://www.tedpavlic.com/teaching/osu/ece209/. Source code at http://hg.tedpavlic.com/ece209/.

So if we know **both** the angle of the major axis of the Lissajous curve **and** the direction of the curve's rotation, then we can determine the quadrant of the phase shift δ. That is,

$$\begin{cases} \delta = 0^\circ & \text{if } \textbf{\textit{line}} \text{ with } \textbf{positive slope} \\ 0^\circ > \delta > -90^\circ & \text{if } \textbf{counter clockwise} \text{ and } \textbf{positive slope} \\ \delta = -90^\circ & \text{if } \textbf{counter clockwise } \textbf{\textit{circle}} \\ -90^\circ > \delta > -180^\circ & \text{if } \textbf{counter clockwise} \text{ and } \textbf{negative slope} \\ \delta = -180^\circ & \text{if } \textbf{\textit{line}} \text{ with } \textbf{negative slope} \\ -180^\circ > \delta > -270^\circ & \text{if } \textbf{clockwise} \text{ and } \textbf{negative slope} \\ \delta = -270^\circ & \text{if } \textbf{clockwise } \textbf{\textit{circle}} \\ -270^\circ > \delta > -360^\circ & \text{if } \textbf{clockwise} \text{ and } \textbf{positive slope} \end{cases} \tag{2}$$

where we consider only *negative* δ because physical systems are casual and will only contribute *delay.* When $\delta = 0^\circ$, the input and output are said to be "in phase." Alternatively, when $\delta = -180^\circ$, the input and output are inverted copies of each other and are said to be "out of phase" or simply "inverted." In the other two cases, when $\delta = -90^\circ$ or $\delta = -270^\circ$, the input and output are said to be "in **quadrature**" (i.e., they are a quarter wavelength away from being in phase). Quadrature motion is perfectly circular and has a wide range of applications throughout engineering.

Finding phase shift from measurements: To determine the precise phase shift δ from measurements, we must use Equation (1). If we know the sinusoidal amplitude A and a measurement $(x(t_0), y(t_0))$ from time t_0, then we can use $x(t_0)$ to solve for $2\pi f t_0 + \phi$, and then we can use $y(t_0)$ to solve for δ. That is,

$$\delta = \arcsin\left(\frac{x(t_0)}{A}\right) - \arcsin\left(\frac{y(t_0)}{A}\right). \tag{3}$$

Because each arcsin can match as many as **two** angles in any 360° range, there are four possible δ — one for each of the four quadrants. So we use Equation (2) to pick the correct δ out of the four.

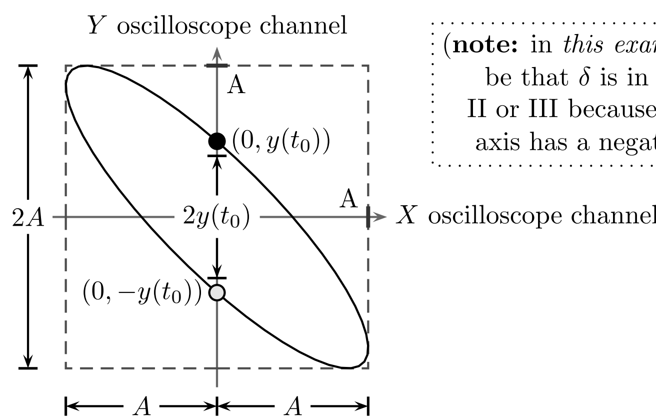
Simple method for the laboratory: The following procedure helps prevent measurement errors from nonzero DC offset. If a measurement at time t_0 has $x(t_0) = 0$, then Equation (3) becomes $\delta = \arcsin(0) - \arcsin(y(t_0)/A)$. This case corresponds to finding the point where the Lissajous figure intersects with the vertical axis.

Your calculator gives you the principal Arcsin.

Let $\theta \triangleq \text{Arcsin}\left(\frac{2y(t_0)}{2A}\right)$.

Then $0^\circ \leq \theta \leq 90^\circ$, and

$$\delta = \begin{cases} -360^\circ + \theta & \text{(Q-I), or} \\ -180^\circ - \theta & \text{(Q-II), or} \\ -180^\circ + \theta & \text{(Q-III), or} \\ -\theta & \text{(Q-IV).} \end{cases}$$



(**note:** in *this example,* it must be that δ is in quadrant II or III because the major axis has a negative slope)

1. Use X **cursors** and X position knob to *horizontally* center the Lissajous figure on the on-screen axes.
2. Use Y **cursors** to measure the *distance* (i.e., ΔY) between two intersection points (i.e., find $2y(t_0)$).
3. Use Y **cursors** to measure the maximum vertical span (i.e., $2A$).
4. Let $\theta \triangleq \text{Arcsin}(\,2y(t_0)/(2A)\,)$ and choose $\delta \in \{-\theta, -180^\circ + \theta, -180^\circ - \theta, -360^\circ + \theta\}$ using Equation (2).
   - Our phase-shifting circuit delays by no more than 180°, and so δ is in quadrant III or IV. Further, the major axis in this example has a negative slope, and so δ is quadrant III (i.e., $\delta = -180^\circ + \theta$).